

Poisson structure and stability analysis of a coupled system arising from the supersymmetric breaking of Super KdV

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Abstract

The Poisson structure of a coupled system arising from a supersymmetric breaking of N=1 Super KdV equations is obtained. The supersymmetric breaking is implemented by introducing a Clifford algebra instead of a Grassmann algebra. The Poisson structure follows from the Dirac brackets obtained by the constraint analysis of the hamiltonian of the system. The coupled system has multisolitonic solutions. We show that the one soliton solutions are Liapunov stable.

Keywords: partial differential equations, supersymmetry, integrable systems, symmetry and conservation laws, Lagrangian and Hamiltonian approach

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1 Introduction

It is well known that the Korteweg-de Vries (KdV) equation has two hamiltonians and associated Poisson structures. The first hamiltonian structure was obtained directly in the framework of the field u satisfying the KdV equation. The second hamiltonian structure was obtained on the framework of the Miura equation (MKdV) and is related to the KdV equation via a Miura transformation. Although both hamiltonian structures may

be formulated in terms of the u field, the origin of the second hamiltonian structure relays on the MKdV formulation. Many extensions of the KdV equation in terms of non-linear coupled partial differential equations for two or more commutative fields have been proposed in the literature [1, 2, 3, 4, 5, 6]. A particularly interesting case is the $N = 1$ super KdV system [7], which has only one local hamiltonian structure. In this case the fields take values on the even and odd part of a Grassmann algebra. When this algebra has finite number of generators, the supersymmetric system may be formulated directly in terms of the real coefficients of a field expansion in terms of the basis of the algebra. One ends up with a coupled system in terms of commutative real fields. The supersymmetric transformation may then be defined directly as a transformation on these real fields which leaves invariant the coupled system.

Recently using such bosonization approach [8] exact solutions of $N = 1$ SKdV equations were obtained [9, 10].

An important aspect of the analysis of the supersymmetric models is its behaviour under supersymmetric breaking conditions. We consider here a supersymmetry breaking implemented by replacing the Grassmann algebra by a Clifford one. This scheme has already been followed in several works, see for example [11].

We consider then a coupled system with fields valued on a Clifford algebra. We find the hamiltonian for such system as well as its Poisson structure. The hamiltonian formulation on phase space presents second class constraints. We then introduce Dirac brackets on the constrained submanifold to define the Poisson structure for such system. The hamiltonian of this system is bounded from below which ensures the a physical admissible content. The solutions of the system satisfy an a priori bound which suggests the existence of global solutions to the system.

Although the system has multisolitonic solutions it does not have an infinite number of local conserved quantities.

We finally consider the stability analysis of the one-soliton solutions of the extended KdV system arising from $N = 1$ SKdV under supersymmetric breaking conditions. Following the approach introduced by [12] and [13] we show that the one-soliton solutions are Liapunov stable, for details see [14].

2 Breaking supersymmetry on SKdV

The $N = 1$ SKdV equations have an infinite sequence of local conserved quantities [7] as well as an infinite sequence of non-local conserved quantities [15, 16, 17]. To break supersymmetry we consider fields u and ξ valued on a Clifford algebra instead of being Grassmann algebra valued. We thus take u to be a real valued field while ξ to be an expansion in terms of the generators $e_i, i = 1, \dots$ of the Clifford algebra:

$$\xi = \sum_i \varphi_i e_i + \sum_{ij} \varphi_{ij} e_i e_j + \sum_{ijk} \varphi_{ijk} e_i e_j e_k + \dots \quad (1)$$

where

$$e_i e_j + e_j e_i = -2\delta_{ij} \quad (2)$$

and $\varphi_i, \varphi_{ij}, \varphi_{ijk}, \dots$ are real valued functions. We define $\bar{\xi} = \sum_{i=1}^{\infty} \varphi_i \bar{e}_i + \sum_{ij} \varphi_{ij} \bar{e}_j \bar{e}_i + \sum_{ijk} \varphi_{ijk} \bar{e}_k \bar{e}_j \bar{e}_i + \dots$ where $\bar{e}_i = -e_i$. We denote as in superfield notation the body of the expansion those terms associated with the identity generator and the soul the remaining ones. Consequently the body of $\xi \bar{\xi}$, denoted by $\mathcal{P}(\xi \bar{\xi})$, is equal to $\sum_i \varphi_i^2 + \sum_{ij} \varphi_{ij}^2 + \sum_{ijk} \varphi_{ijk}^2 + \dots$. In what follows, without loss of generality, we rewrite $\mathcal{P}(\xi \bar{\xi}) = \sum_i \varphi_i^2 + \sum_{ij} \varphi_{ij}^2 + \sum_{ijk} \varphi_{ijk}^2 + \dots$ simply as $\mathcal{P}(\xi \bar{\xi}) = \sum_i \varphi_i^2$.

The resulting system after the breaking of supersymmetry becomes

$$\begin{aligned} u_t &= -u''' - uu' - \frac{1}{4}(\mathcal{P}(\xi \bar{\xi}))' \\ \xi_t &= -\xi''' - \frac{1}{2}(\xi u)' \end{aligned} \quad (3)$$

This coupled KdV system arises from the following Lagrangian, expressed in terms of the fields w and η_i related to u and φ_i by

$$u = w' \text{ and } \varphi_i = \eta'_i, \quad (4)$$

$$S(w, \eta_i) \equiv \int dx dt \left[\frac{1}{2} w' \partial_t w + \frac{1}{6} (w')^3 - \frac{1}{2} (w'')^2 + \frac{1}{4} w' (\eta'_i)^2 + \frac{1}{2} \eta'_i \partial_t \eta_i - \frac{1}{2} (\eta''_i)^2 \right], \quad (5)$$

where a repeated index i implies summation on that index. Variations of the action $S(w, \eta_i)$ yields the corresponding field equations (3).

The hamiltonian and corresponding Poisson structure associated to the action (5) follows by introducing the conjugate momenta associated to w, η_i . We will denote them by p, σ_i respectively

$$\begin{aligned} p &:= \frac{\partial \mathcal{L}}{\partial(\partial_t w)} = \frac{1}{2} w' = \frac{1}{2} u \\ \sigma_i &:= \frac{\partial \mathcal{L}}{\partial(\partial_t \eta_i)} = \frac{1}{2} \eta'_i = \frac{1}{2} \varphi_i. \end{aligned} \quad (6)$$

(6) are primary constraints on the phase space. The hamiltonian may be obtained by performing a Legendre transformation.

We obtain

$$H = \int_{-\infty}^{+\infty} \left(-\frac{1}{3} u^3 - \frac{1}{2} u \mathcal{P}(\xi \bar{\xi}) + (u')^2 + \mathcal{P}(\xi' \bar{\xi}') \right) dx. \quad (7)$$

Following the Dirac approach for constrained systems it turns out that (6) are the only constraints on the phase space, and they are second class constraints. The Poisson structure on the constrained submanifold of phase space is then given by the Dirac brackets. They are given by

$$\begin{aligned} \{u(x), u(y)\}_{DB} &= \partial_x \delta(x, y), \\ \{\varphi_i(x), \varphi_j(y)\}_{DB} &= \delta_{ij} \partial_x \delta(x, y), \\ \{u(x), \varphi_i(y)\}_{DB} &= 0. \end{aligned} \quad (8)$$

We thus have obtained the hamiltonian H and the Poisson structure of the coupled KdV system (3). Besides the hamiltonian H there are other conserved quantities under the evolution determined by the coupled KdV system (3). The following are conserved quantities

$$\begin{aligned}\hat{H}_{\frac{1}{2}} &= \int_{-\infty}^{+\infty} \xi dx, \\ \hat{H}_1 &= \int_{-\infty}^{+\infty} u dx, \\ V \equiv \hat{H}_3 &= \int_{-\infty}^{+\infty} (u^2 + \mathcal{P}(\xi\bar{\xi})) dx.\end{aligned}\tag{9}$$

It is interesting to notice that the non-local quantity

$$\int_{-\infty}^{\infty} \xi(x) \int_{-\infty}^x \xi(s) ds dx \tag{10}$$

is also conserved. Remarkably, this quantity in terms of a Grassmann valued ξ is also conserved for the $N = 1$ SKdV system.

The system (3) besides being not invariant under supersymmetry, by construction, it does not have an infinite sequence of local conserved quantities. It does have, however, multisolitonic solutions. We discuss now the stability properties of the one-soliton solutions.

3 Stability of the solitonic solutions

An important first step in the analysis is to show an a priori bound [18] for the solutions of the system (3).

We denote by $\| \cdot \|_{H_1}$ the Sobolev norm

$$\|(u, \xi)\|_{H_1}^2 = \int_{-\infty}^{+\infty} \left[(u^2 + \sum_{i=0}^{\infty} \varphi_i^2) + (u'^2 + \sum_{i=0}^{\infty} \varphi_i'^2) \right] dx. \tag{11}$$

We obtain from the expressions of V and H

$$\|(u, \xi)\|_{H_1}^2 \leq V + H + \frac{1}{2} \int_{-\infty}^{+\infty} |u| (u^2 + \mathcal{P}(\xi\bar{\xi})) dx. \tag{12}$$

We may now use

$$\sup |u| \leq \frac{1}{\sqrt{2}} \|u\|_{H_1} \leq \frac{1}{\sqrt{2}} \|(u, \xi)\|_{H_1} \tag{13}$$

to get

$$\|(u, \xi)\|_{H_1}^2 \leq V + H + \frac{1}{2\sqrt{2}} V \|(u, \xi)\|_{H_1}. \tag{14}$$

It then follows

$$\|(u, \xi)\|_{H_1} \leq \frac{d + \sqrt{d^2 + 4e}}{2} \tag{15}$$

where $d = \frac{1}{2\sqrt{2}}V$ and $e = V + M$. We notice that $d^2 + 4e \geq 0$.

Given V and H from the initial data and a solution satisfying those initial conditions, then $\|(u, \xi)\|_{H_1}$ is bounded by (15) for all $t \geq 0$. This is an a priori bound and a strong evidence of the existence of the solution for $t \geq 0$.

We consider the stability of a solution $(\hat{u}, \hat{\xi})$ of (3) in the Liapunov sense:

$(\hat{u}, \hat{\xi})$ is stable if given ϵ there exists δ such that for any solution (u, ξ) of (3), satisfying at $t = 0$

$$d_I \left[(u, \xi), (\hat{u}, \hat{\xi}) \right] < \delta \quad (16)$$

then

$$d_{II} \left[(u, \xi), (\hat{u}, \hat{\xi}) \right] < \epsilon \quad (17)$$

for all $t \geq 0$.

We consider first the stability of the ground state solution $\hat{u} = 0, \hat{\xi} = 0$. We take d_I and d_{II} to be the Sobolev norm $\|(u - \hat{u}, \xi - \hat{\xi})\|_{H_1}$.

We get

$$V \leq \|(u, \xi)\|_{H_1}^2$$

and

$$H \leq \int_{-\infty}^{+\infty} \left(\frac{1}{2}|u| (u^2 + \mathcal{P}(\xi\bar{\xi})) + u'^2 + \mathcal{P}(\xi'\bar{\xi}') \right) dx \leq \frac{1}{2\sqrt{2}} \|(u, \xi)\|_{H_1}^3 + \|(u, \xi)\|_{H_1}^2.$$

It then follows from the a priori bound (15) the stability of the ground state $\hat{u} = 0, \hat{\xi} = 0$.

We now consider the stability of the one-soliton solution $\hat{u} = \phi, \hat{\xi} = 0$, where ϕ is the one-soliton solution of KdV which satisfies

$$\phi'' + \frac{1}{2}\phi^2 = \mathcal{C}\phi. \quad (18)$$

The proof of stability is based on estimates for the u field which are analogous to the one presented in [12, 13] while a new argument will be given for the ξ field. The distances we will use are

$$d_I [(u_1, \xi_1), (u_2, \xi_2)] = \|(u_1 - u_2, \xi_1 - \xi_2)\|_{H_1} \quad (19)$$

$$d_{II} [(u_1, \xi_1), (u_2, \xi_2)] = \inf_{\tau} \|(\tau u_1 - u_2, \xi_1 - \xi_2)\|_{H_1} \quad (20)$$

where τu_1 denotes the group of translations along the x -axis. d_{II} is a distance on a metric space obtained by identifying the translations of each $u \in H_1(\mathbb{R})$ [12]. d_{II} is related to a stability in the sense that a solution u remains close to $\hat{u} = \phi$ only in shape but not necessarily in position.

At $t = 0$ we assume that

$$d_I [(u, \xi), (\phi, 0)] = \|(h, \xi)\|_{H_1} < \delta. \quad (21)$$

Using estimates as in [12, 13] one can show that

$$|\Delta H| \leq \left[\max(1, \mathcal{C}) + \frac{1}{3\sqrt{2}}\delta \right] \| (h, \xi) \|_{H_1}^2 \quad (22)$$

where $h \equiv u - \phi$.

The next argument introduces besides the ideas in [12, 13], new estimates on ξ , see [14]. After several delicate estimates we obtain

$$\Delta H \geq \frac{1}{6} l \{d_{II} [(u, \xi), (\phi, 0)]\}^2 \quad (23)$$

where $l = \min(1, \mathcal{C})$.

The upper and lower bounds for ΔH imply the stability of the one-soliton solution of the coupled KdV system.

4 Conclusions

By breaking the supersymmetry in the $N = 1$ SKdV equations we arrive to a coupled KdV system. We found its hamiltonian and associated Poisson structure. The hamiltonian is bounded from below and consequently has an admissible physical interpretation. This important property is then used to show the stability of the ground state solution as well as the one-soliton solutions of the coupled KdV system. We determine also an a priori bound giving a strong evidence of the existence of solutions for all $t \geq 0$.

The property of having a hamiltonian bounded from below is very important and it is shared with the hamiltonian of the super KdV equations. However, the stability of the solitonic solutions of $N = 1$ super KdV equations has, so far, not been proven in the literature.

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